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An Asset Allocation Primer: Connecting Markowitz, Kelly and Risk Parity

AUTHORS

Jamil Baz
Managing Director
Global Head of Client Analytics

Helen Guo
Vice President
Quantitative Research Analyst

As curious as it may sound, few asset allocation primers describe portfolio models while connecting them to each other. This article does so by describing and contrasting the mechanics of standard asset allocation models, including the utility-based, Markowitz, Kelly, risk parity and fixed allocation approaches. It seeks to accomplish four objectives: First, it develops the mathematics of the models; second, it identifies the precise conditions under which the models are equivalent; third, it discusses the specifics – risk/return characteristics – of each approach; and last, it provides numerical examples that compare allocations to asset classes and return and risk measures for various models.

The main points are these:

- Utility-based asset allocation models are the most general class of models.
- For some specific utility functions and return distributions, the Markowitz approach is a special example of a utility-based model.
- The Kelly framework can, in turn, be a special example of the Markowitz model, for a specific risk-aversion coefficient.
- Under some conditions on correlations and Sharpe ratios, risk parity is optimal and collapses into a Markowitz model.
- Kelly strategies present unique characteristics as they outperform alternative strategies almost surely over long time periods.
- These unique characteristics come at the cost of generally extreme risk parameters.

- Despite a lack of theoretical foundations, risk parity strategies have been partly fueled by their empirical success. One may call this luck.
- Numerical examples show a wide disparity across strategies.

1. THE UBIQUITOUS MEAN-VARIANCE ANALYSIS

Expected utility maximization and mean-variance analysis (Markowitz 1952, 1959) are two prevailing approaches to asset allocation. Most academic researchers prefer the expected utility maximization approach to model choices under uncertainty because the solutions will satisfy certain basic conditions of rationality (von Neumann and Morgenstern 1944). It is often the ultimate standard for evaluating the optimality of a model or ranking different models.

This section discusses the close connection between the two approaches and shows their convergence in a continuous-time power utility maximization problem with constant investment opportunities. In addition, we identify the assumptions under which the optimal solution coincides with the Kelly (growth-optimal) strategy or the risk parity strategy. We will discuss these two strategies in more detail in Section 2.

1.1 Mean-variance analysis

The mean-variance analysis assumes investors only care about the first two moments of portfolio returns and maximize the mean-variance utility function[1]

$$E(r_p) - \frac{1}{2}\lambda V(r_p),$$

where λ measures the investor's risk aversion. This approach is easy to implement and understand. It is by far the most prominent asset allocation model for practitioners and underlies most of the products and advice provided by financial institutions.

For single-period portfolio optimizations, mean-variance analysis is consistent with the theoretically popular expected utility maximization framework if the investor's utility is quadratic or if returns are normally distributed. Even if the utility function is not quadratic, Levy and Markowitz (1979) showed that mean-variance optimization is equivalent to maximizing the expectation of the second-order Taylor approximations of standard utility functions, such as the power utility and the exponential utility. Thus, the consistency of the mean-variance analysis with expected utility maximization for single-period optimization problems depends on the degree of non-normality of returns, the investment horizon and the specific functional form of the investor's utility.

For multiperiod portfolio optimizations, maximizing the commonly preferred constant relative risk aversion (CRRA) utility with continuous rebalancing[2] and lognormal prices will lead to mean-variance optimal (MVO) solutions, as discussed below.

The CRRA utility (also known as the power utility or isoelastic utility) has the following functional form:

$$U(W) = \frac{W^{1-\gamma}-1}{1-\gamma} \qquad (\gamma > 0),$$

where γ measures the constant relative risk aversion of the investor. In the extreme case where γ=0, the investor is risk neutral. When γ=1, the investor has a log utility.[3] Empirical studies and surveys suggest most investors' relative risk aversion to wealth is between 1 and 10 (Ang 2014). The most commonly accepted values for asset allocation are between 1 and 5 and centered around 3.

A nice feature of the CRRA utility is that it leads to optimal portfolio allocations not dependent on the level of wealth. This wealth homogeneity property is important for the scalability of investment management because it allows the same model to be used for different sizes of assets.

The investor with wealth W_0 at time 0 maximizes the expected utility of his or her final wealth at time T by choosing the optimal portfolio weights in N risky asset(s) and a risk-free asset with continuous rebalancing. For simplicity, assume the investor has no labor income or intermediate consumptions and there are no transaction costs.

Let's start with a simple case with only one risky asset. Assume the risky asset's price follows a standard geometric Brownian motion with parameters (μ,σ^2) and the risk-free asset has a geometric growth rate of r. Assume asset returns are independent and identically distributed (i.i.d.) over time.

With the standard intertemporal budget constraint and Itô's lemma, it can be shown (see Appendix 2) that the optimal allocation to the risky asset, x^*, is the solution to the following problem:

$$\max_x \left[r + x(\mu - r) - \frac{1}{2}\gamma x^2\sigma^2\right]T.$$

Therefore, the solution of the optimal weight to the risky asset is

$$x^* = \frac{\mu - r}{\gamma\sigma^2} = \frac{s}{\gamma\sigma},$$

where s is the instantaneous Sharpe ratio of the risky asset.

Note that the immediate objective function (the instantaneous integrand) and the optimal solution resemble those under the mean-variance optimization framework where the risk aversion parameter of the CRRA utility function γ coincides with the risk aversion parameter for mean-variance optimization λ. It is the continuous-time instantaneous equivalent of the static single-period Markowitz model. The optimal allocation to the risky asset is directly proportional to the excess return of the risky asset and inversely proportional to its variance or the investor's relative risk aversion.

It is straightforward to extend the analysis to the cases where there are two or more risky assets following a multivariate geometric Brownian motion with parameters (μ,Σ). The optimal allocation to the N risky assets in this case is:

$$x^* = \frac{1}{\gamma}\Sigma^{-1}(\mu - r).$$

The composition of the subportfolio of risky assets does not depend on the investor's risk aversion. It also achieves the maximum possible Sharpe ratio given the investment universe. The risk aversion parameter γ only affects the total weight in risky assets and the weight in the risk-free asset. This is similar to the two-fund separation theorem in the traditional mean-variance analysis.

1.2 Kelly and risk parity as special cases

It is easy to see that the Kelly strategy is a special case of the optimal solution to the CRRA utility maximization problem, with a specific value for the risk aversion parameter. If γ=1, the CRRA utility is reduced to log utility. The solution will maximize the expected geometric growth rate of the portfolio and therefore is often referred to as the optimal growth portfolio (or the Kelly portfolio).

It is less straightforward to find the conditions under which the optimal solution will be a risk parity portfolio. Let's start from the simple case where N=2. We can write the optimal weights to the two risky assets as functions of their volatilities, Sharpe ratios and correlation ρ:

$$x^* = \frac{1}{\gamma(1-\rho^2)}\begin{pmatrix} \frac{1}{\sigma_1}(s_1 - \rho s_2) \\ \frac{1}{\sigma_2}(s_2 - \rho s_1) \end{pmatrix}.$$

If the two risky assets have the same Sharpe ratios, the optimal risky asset portfolio is also a risk parity portfolio in the sense of both equal stand-alone risks $(x_1\sigma_1 = x_2\sigma_2)$ and equal risk contributions $(x_1{}^2\sigma_1{}^2 + x_1x_2\rho\sigma_1\sigma_2 = x_2{}^2\sigma_2{}^2 + x_1x_2\rho\sigma_1\sigma_2)$ where risk is measured by volatility.

When $N > 2$, if the assets have the same Sharpe ratios and the correlations are constant, the optimal allocation will still be a risk parity portfolio (see Appendix 3).

2. MORE ON KELLY AND RISK PARITY

The Kelly strategy and the risk parity strategy are two popular alternative asset allocation models in the investment industry. In Section 1, we show the conditions under which the solution to a CRRA utility maximization problem coincides with these two strategies, respectively. In this section, we discuss the features of the two models in more detail.

2.1 The Kelly (growth-optimal) strategy

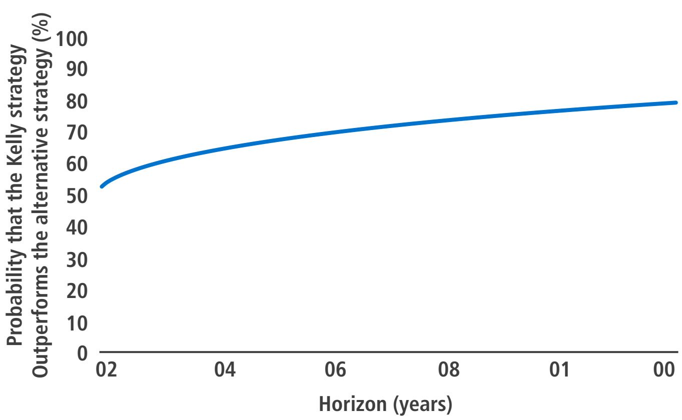
The Kelly (or growth-optimal) strategy maximizes the expected geometric growth rate of a portfolio. This is equivalent to maximizing the expected logarithm of final wealth. Therefore, the optimal dynamic portfolio choice is myopic. The Kelly strategy allows the breakdown of a multiperiod optimization problem into single-period problems that are easy to solve.

In addition to its convenience of implementation, the Kelly strategy has a few other impressive properties. For example, it minimizes the expected time to reach a given wealth target. In the long run, the Kelly strategy asymptotically dominates all essentially different strategies (Breiman 1961).

Another interesting property of the Kelly strategy is its competitive optimality (Bell and Cover 1980). Denote an investor's wealth streams under the Kelly strategy and under any other alternative strategy by W_t^K and W_t^A, respectively. It can be shown that $E(\frac{W_t^A}{W_t^K}) \leq 1$ for any t. It means the ratio of the investor's wealth associated with any competing strategy to the wealth associated with the Kelly strategy is expected to be less than 1 for any horizon.

In our previous example of a single risky asset and continuous rebalancing, assume the risky asset has an annual risk premium of 3% and an annual volatility of 15%. Consider the Kelly strategy (x=1.3) and an alternative fixed mix strategy where x=0.2. Exhibit 1 shows the probability that the Kelly strategy will outperform the alternative strategy over time. In fact, this probability is always higher than 50% for any alternative fixed mix strategy and converges to 100% as the investment horizon goes to infinity (see Appendix 4).

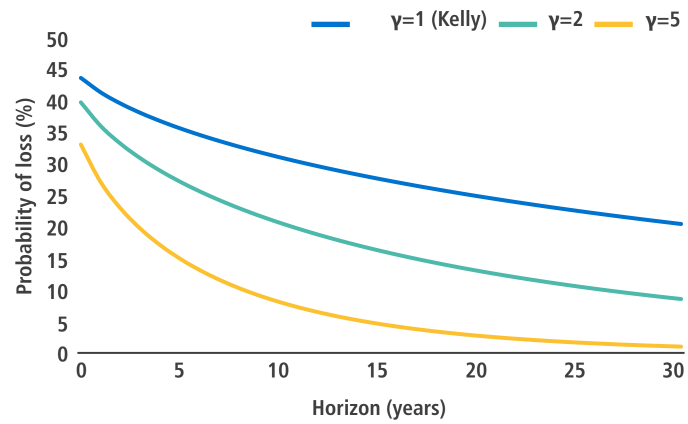
Exhibit 1: Probability that the Kelly strategy will outperform another fixed mix strategy



Source: PIMCO. **Hypothetical example for illustration purposes only.** See Appendix 4 for more details.

The main disadvantage of the Kelly strategy is that it can be very risky in the short run. This is consistent with the fact that the log utility is a special case of the CRRA utility when the risk aversion parameter γ is equal to 1, the lower bound of its plausible values. Consider the same simple example with one risky asset, as in Exhibit 1, where $r = 1\%$, $\mu = 4\%$ and $\sigma = 15\%$. Exhibit 2 shows the probability of loss for the optimal portfolio over time under different risk aversion parameters. The probability of loss is decreasing in both T and γ. The Kelly strategy (γ=1) has a significantly higher risk of loss than the other optimal strategies, with γ=2 and γ=5.

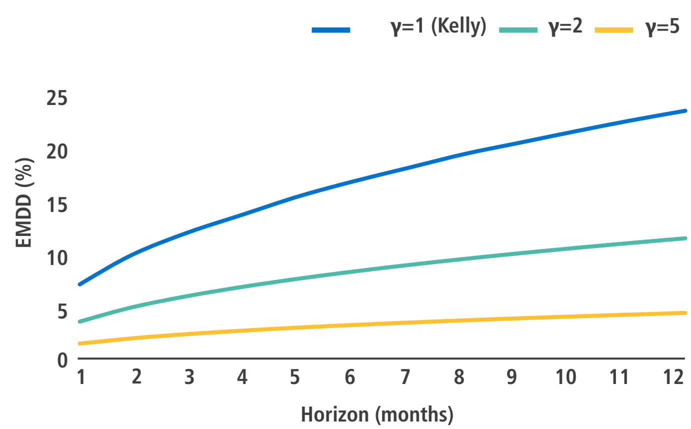
Exhibit 2: Probability of loss for optimal strategies with different risk aversion parameters



Source: PIMCO. **Hypothetical example for illustration purposes only.** See Appendix 2 (Case 1) for more details.

Another risk measure in which an investor might be interested is expected maximum drawdown (EMDD), which measures the expected maximum percentage loss from peak to trough within a specific investment horizon. Exhibit 3 plots EMDD for optimal strategies with different risk aversion parameters over horizons up to 12 months, based on the same parameter assumptions as in the previous exhibits. For any given horizon, EMDD is decreasing in the risk aversion parameter, which means the Kelly strategy is the riskiest based on this measure.

Exhibit 3: EMDD for optimal strategies with different risk aversion parameters



Source: PIMCO. **Hypothetical example for illustration purposes only.** See Appendix 5 for more details.

In addition, the minimal risk aversion implies the optimal portfolio choice is more sensitive to estimation errors for the return distributions because the weight in the risky assets is inversely proportional to γ. In practice, Kelly proponents often apply a fractional Kelly strategy (investing a fixed fraction of money in the Kelly strategy and the rest in cash) to reduce risk and protect against estimation errors. However, in the CRRA utility maximization problem discussed previously, $x^* = \frac{1}{\gamma}\Sigma^{-1}(\mu - r)$, meaning the risk aversion parameter γ only affects the weights in the risky assets versus cash, not the composition of the risky subportfolio. Therefore, applying a fractional Kelly strategy is equivalent to using a γ higher than 1 in the CRRA utility maximization/mean-variance optimization problem. A fractional Kelly strategy is simply a mean-variance solution with a potentially more realistic risk aversion parameter.

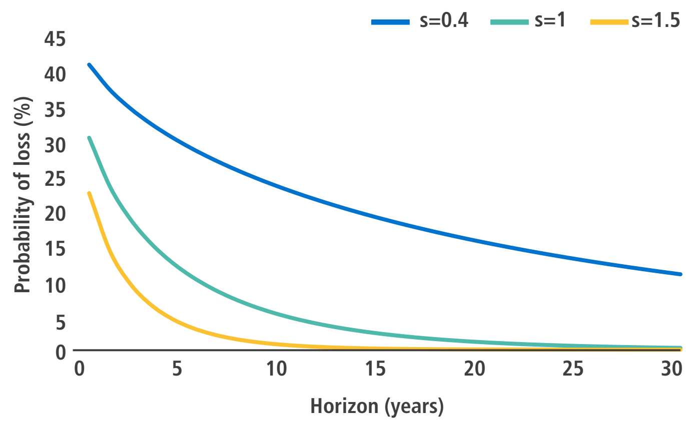
Although Kelly dominates other strategies in the long run, it can take a very long time for this to happen. In the example shown in Exhibit 1, it would take 227 years to have a 90% confidence that the Kelly strategy (x=1.3) will outperform the alternative fixed mix strategy with x=0.2.

Next we compare Kelly strategies under different investment opportunities by fixing γ=1 and changing the instantaneous Sharpe ratio of the risky asset. As shown in Appendix 2 (Case 1), the value of the Kelly portfolio follows:

$\ln\left(\frac{W_T{}^*}{W_0}\right) = (r + \frac{1}{2}s^2)T + s\sqrt{T}Z$, where $Z \sim N(0,1)$.

This means the annual volatility of the Kelly strategy is equal to s, the instantaneous Sharpe ratio of the risky asset. Therefore, the volatility of the Kelly strategy is higher when the investment opportunity is better. However, higher volatility does not necessarily mean a higher probability of loss, because the expected return of the Kelly strategy is also higher, with a higher Sharpe ratio of the risky asset. A better investment opportunity is a mixed blessing for Kelly strategies in terms of probability of loss. Appendix 2 shows the probability of loss for a Kelly strategy is increasing in s if $s^2<2r$ and decreasing in s if $s^2>2r$. Given the current low-interest-rate environment, an improved Sharpe ratio most likely implies a lower probability of loss, as illustrated in Exhibit 4, with the assumption that $r = 1\%$.

Exhibit 4: Probability of loss for Kelly strategies under different investment opportunities



Source: PIMCO. **Hypothetical example for illustration purposes only.** See Appendix 2 for more details.

Ziemba (2015) characterized Kelly portfolios in reality as having high concentrations in very few investments, frequent and big short-term losses and superior long-term growth, and provided examples of highly successful investors who are known to follow or likely follow the Kelly strategy, based on observed characteristics of their investments. If the quality of the inputs (estimated return distributions – especially the means) is high and the investment horizon is long enough to allow compounding of many rebalancing periods, the Kelly strategy can be a powerful asset allocation and risk management tool that potentially maximizes long-term wealth growth. However, for short-term investors who have more risk aversion than that implied by log utility, the Kelly strategy tends to result in more risk than they might be willing to take under general investment conditions. It should be considered with great caution.

2.1.1 Volatility pumping and portfolio growth

Volatility pumping is a strategy or mechanism often associated with growth-optimal portfolios. Luenberger (1998) coined the term and described how rebalancing a portfolio to fixed weights can generate a higher expected growth rate than the weighted average of the expected growth rates of the individual assets. He attributed the excess growth to the pumping action that automatically "buys low and sells high" through the process of rebalancing,[4] even when returns are intertemporally independent. The implications are that volatility is opportunity and investors should seek it out rather than shunning it.

We believe the primary source of the excess growth rate of the portfolio in the case of i.i.d. returns is not rebalancing itself but variance reduction relative to the mean. To see this, we need to understand the relationship between the expected log returns (geometric growth rates) and expected arithmetic returns:

$$E(\ln(1+r)) \approx E(r) - \frac{1}{2(1+E(r))^2} Var(r).$$

This says an asset's expected log return is lower than its expected arithmetic return by approximately half of the variance of the arithmetic return. This relationship holds exactly in continuous time for assets following geometric Brownian motions and approximately in short discrete-time intervals or for assets following other distributions (see Appendix 6).

A popular and simple example of volatility pumping is to generate a positive expected geometric growth rate for a portfolio with two assets that have zero expected geometric growth rates. One asset is risky and follows a standard geometric Brownian motion with parameters (μ,σ^2). By construction, we have $\mu - \frac{1}{2}\sigma^2 = 0$ (for example, 2% mean and 20% volatility). The other asset is risk-free with a zero growth rate. Then the value of a continuously rebalanced portfolio with fixed weight x in the risky asset and $1-x$ in the risk-free asset will follow a geometric Brownian motion with parameters $(x\mu, x^2\sigma^2)$. Therefore, the expected geometric growth rate of the portfolio is:

$$x\mu - \frac{1}{2}x^2\sigma^2 = x\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{1}{2}x(1-x)\sigma^2 = \frac{1}{2}x(1-x)\sigma^2.$$

Therefore, the expected excess geometric growth rate of the portfolio is positive for any $x \in (0,1)$ and is maximized at $\frac{1}{8}\sigma^2$ when x=0.5 under the Kelly strategy in this simple two-asset example. A Kelly strategy is not required to achieve a positive excess geometric growth rate for the portfolio.

The excess geometric growth rate is increasing in the volatility of the risky asset; this is sometimes used as evidence that investors should seek out volatile assets to gain an excess growth rate by applying a volatility pumping strategy. However, even without rebalancing, the excess growth rate can still be achieved for relatively short investment horizons because the relationship between expected log return and expected arithmetic return still holds, approximately, in this case. Therefore, the excess growth rate is not the direct result of rebalancing itself, but a result of variance reduction more than the mean reduction when the risky asset is combined with the risk-free asset with weights between 0 and 1 (the mean is scaled by x, and the variance is scaled by x^2). Even though rebalancing involves selling the risky asset after its price goes up and buying it after its price goes down, it does not redistribute wealth to the asset with better future expected growth because returns are intertemporally independent and both assets have zero expected growth.

For longer horizons without rebalancing (i.e., a long-term buy-and-hold strategy), the portfolio weights will drift away from the optimal value over time and therefore reduce the expected growth rate of the portfolio. The main contribution of rebalancing to portfolio growth is to keep the weights at the optimal level that maximizes portfolio growth over time rather than buying low and selling high, unless the returns exhibit certain intertemporal dependence, such as mean reversion.

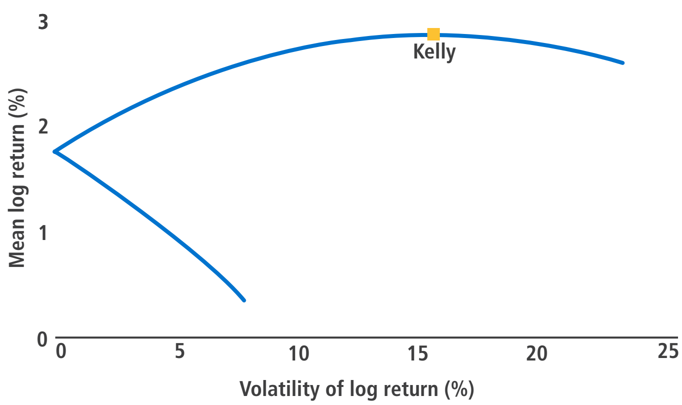
2.1.2 The growth efficient frontier

We have shown that the Kelly strategy maximizes the expected log return or geometric growth rate of the portfolio. Although the strategy dominates other strategies in the long run, it is often criticized for being too risky in the short run because it does not take into account the volatility of the log return or the geometric growth rate.

In practice, many long-term investors do care about the volatility of the log return in addition to the mean log return. Luenberger (1993) proposed that an investor who considers only long-term performance will evaluate a portfolio only on the basis of the mean and variance of its single-period log return given i.i.d. investment opportunities over time.

It is natural to ask whether we can have risk/return trade-off where returns are measured in terms of log returns, similar to the simple risk/return trade-off in the traditional mean-variance analysis. Exhibit 5 plots such a growth efficient frontier based on the model assumptions for the numerical example in Section 3.

Exhibit 5: Growth efficient frontier for three risky assets and a risk-free asset



Source: PIMCO

Compared with the traditional mean-variance efficient frontier with a risk-free asset, which is a straight line passing the risk-free asset and the maximum Sharpe ratio portfolio, the growth efficient frontier is a curve with a maximum mean log return point, which is the Kelly strategy. The frontier is very flat near the maximum point, meaning that reducing the mean log return slightly can greatly reduce its volatility.

In the CRRA utility maximization problem with assets following geometric Brownian motion presented above, all the portfolios on the growth efficient frontier can be constructed by mixing the risk-free asset and the log-optimal portfolio. Under this model setup, fractional Kelly is efficient.

2.2 Risk parity

Risk-based portfolio construction techniques such as risk parity have gained great popularity among investors and academics alike since the 2008 global financial crisis. The demand stems from the higher risk aversion or awareness of investors, as well as the lack of robustness of the traditional mean-variance optimal portfolio due to its sensitivity to the estimation errors in its inputs, especially in expected returns. For risk parity and similar risk-based approaches, such as minimum-variance portfolios, the only input required is the risk estimation, which is generally believed to be more robust than return forecasts (see, for example, Merton 1980).

Despite the vast and still-growing literature on risk parity, there is no consensus on the exact definition of a risk parity portfolio. The general philosophy is to equalize risk from different portfolio components. The risk can be the components' stand-alone volatilities, ignoring correlations among component returns or their contributions to portfolio volatility. An equal stand-alone risk (ESR) portfolio equalizes the products of the portfolio weights in the components and the volatilities of those components. Therefore, the risk parity weights are inversely proportional to the component return volatilities: $x_i = \frac{\sigma_i^{-1}}{\sum_j \sigma_j^{-1}}$. An equal risk contribution (ERC) portfolio equalizes the portfolio volatility contribution from each component: $RC_i = x_i \frac{(\Sigma x)_i}{\sqrt{x^T \Sigma x}}$, where x is the vector of portfolio weights in the components, the subscript represents the ith element of a vector and Σ is the covariance matrix of component returns. An alternative interpretation of the ERC portfolio, proposed by Lee (2011), is that the components' weights are inversely proportional to their betas with respect to the portfolio. The ERC portfolio generally has no analytical solutions and has to be solved for numerically. In the special case of constant pairwise correlations, the two portfolios coincide.

The portfolio components can be defined based on the asset classes, securities or risk factors that are the underlying drivers of risk.[5] In addition, because there is no theory to guide these decisions, diverse risk parity strategies can be constructed by varying investment universe, grouping schemes and investment horizons.

The lack of a standard definition for risk parity has stymied efforts to construct a benchmark to evaluate the performance of different risk parity strategies. In fact, there has been a considerable disparity in the performance of risk parity funds. Investors need to exercise extra caution in choosing risk parity products: What they think they are getting may not be consistent with the investment objectives they want to achieve, due to the heterogeneity of the actual strategies implemented and the resulting performance divergence.

The increasing popularity of risk parity strategies has been fueled in part by their empirical success over the past few decades. However, history is only one realized path among many possible paths. As the familiar disclaimer says, "Past performance is not a guarantee of future returns." A classic point of contention in regard to risk parity is whether its historical success is simply driven by the long-lasting fixed income rally, which may not persist in the future.

As a heuristic, the risk parity strategy suffers from a lack of a solid theoretical foundation. As discussed previously, its mean-variance optimality depends on strong assumptions on the Sharpe ratios and the correlation matrix of the component returns. There have been some recent attempts to fill this gap. Asness, Frazzini and Pedersen (2012) tried to justify the superiority of risk parity using the leverage aversion theory,[6] which suggests low-beta assets offer higher risk-adjusted returns than high-beta assets due to the leverage aversion of average investors; an investor with lower leverage aversion than average can exploit the risk premium with risk parity portfolios due to their tilts toward safer assets relative to the market portfolio. In another empirical study, however, Anderson, Bianchi and Goldberg (2012) compared the historical performance of unlevered and levered risk parity strategies with 60/40 and value-weighted strategies during 1926-2010 and subperiods and showed that incorporating trading costs can negate the outperformance of risk parity strategies, especially with leverage. Fisher, Maymin and Maymin (2015) provided another perspective to justify risk parity for investors facing uncertainty in the expected return estimates. They described the exact conditions for the uncertainty set under which the minimum Sharpe ratio of the risk parity (defined as ESR) portfolio is higher than those of other portfolios, including the tangency portfolio.

In practice, the leverage required for risk parity portfolios to achieve a typical return or risk target poses potentially higher downside risk when the safer assets, such as credit, have negative skews. Leverage also can exacerbate turnover, leading to much higher trading costs than those for unlevered strategies (Anderson, Bianchi and Goldberg 2012).

Although the implementation of risk parity does not require expected return forecasts, its performance is almost always evaluated based on both return and risk measures. If investors do not believe expected returns can be estimated reliably, are willing to assume the included portfolio components have equal long-term Sharpe ratios and pairwise correlations, and are willing and able to use leverage, risk parity can be a very efficient heuristic for strategic asset allocation. In the more extreme case in which investors also have very low confidence in volatility estimates, even a naive equally weighted portfolio can be a viable heuristic with decent performance (see, for example, DeMiguel, Garlappi and Uppal 2009, and Chaves et al. 2011).

3. NUMERICAL EXAMPLES

Here we present some simple examples related to the models discussed previously. For a more systematic discussion of practical quantitative approaches to asset allocation, we recommend Naik et al. (2016).

Suppose an investor faces the optimization problem described in Section 1. The hypothetical investment universe consists of four assets: equities, bonds, commodities and cash. Their i.i.d. investment opportunities are shown below (see Exhibit 6).

The mean-variance optimal weights to the risky assets are given by: $x^* = \frac{1}{\gamma}\Sigma^{-1}(\mu - r)$. The weight in cash is therefore $1 - \mathbf{1}^T x^*$, which can be either positive (lending) or negative (borrowing). The relative risk aversion parameter γ only affects the total weight in the risky assets and does not change the composition of the optimal subportfolio of risky assets. A risk parity portfolio equalizes the risk contributions from the three risky assets based on the covariance matrix only. For comparison, we also include a traditional portfolio that invests 60% in stocks and 40% in

Exhibit 6: Model assumptions

Asset	μ	σ	Correlation matrix			Instantaneous Sharpe ratio
Equities	3.2%	15.0%	1			0.10
Bonds	2.2%	3.6%	-0.06	1		0.11
Commodities	2.2%	16.9%	0.34	-0.02	1	0.03
Cash	1.8%					

Source: PIMCO. **Hypothetical example for illustrative purposes only.** The distributional assumptions for different assets are based on PIMCO's 10-year capital market assumptions (model-based except for Commodities, which was based on an internal survey of portfolio managers) and long-term risk estimates as of July 2017.

Exhibit 7: Unconstrained MVO, Kelly, unlevered risk parity and 60/40 portfolios

	Asset	CRRA utility maximization (MVO)			Risk parity	60/40
		γ=5	γ=2	γ=1 (Kelly)		
Asset allocation	Equities	14.3%	35.7%	71.4%	15.0%	60.0%
	Bonds	63.2%	158.1%	316.2%	72.1%	40.0%
	Commodities	-0.8%	-2.0%	-4.0%	12.9%	0.0%
	Cash	23.3%	-91.8%	-283.7%	0.0%	0.0%
Risk allocation	Equities	46.1%	46.1%	46.1%	33.3%	98.4%
	Bonds	54.7%	54.7%	54.7%	33.3%	1.6%
	Commodities	-0.8%	-0.8%	-0.8%	33.3%	0.0%
	Cash	0.0%	0.0%	0.0%	0.0%	0.0%
Estimated annual return		2.2%	2.9%	4.1%	2.3%	2.8%
Estimated annual volatility		3.0%	7.5%	15.1%	4.4%	9.0%
Sharpe ratio		0.15	0.15	0.15	0.13	0.11
Expected maximum drawdown (1 Year)		3.0%	8.4%	17.7%	4.6%	10.4%

Source: PIMCO. **Hypothetical example for illustrative purposes only.** Risk is measured by annual volatility. The MVO portfolios are unconstrained in the sense that there are no constraints on short positions in the assets, including cash, and the borrowing rate is assumed to be equal to the risk-free rate.

bonds.[7] Exhibit 7 shows the asset allocation, risk allocation, and return and risk estimates of different models.

Within the group of MVO portfolios, the Kelly strategy is very risky, with extremely high leverage, although its subportfolio of risky assets has the same composition as the MVO portfolios with higher risk aversion parameters.

The ratio between equities and bonds in the MVO portfolios is similar to that in the risk parity portfolio. This is consistent with the assumption that the two assets have similar Sharpe ratios. The assumption for the Sharpe ratio of commodities, however, is much lower. This, together with the different correlations, means the risk parity portfolio will not be mean-variance optimal.

Exhibit 7 also shows the risk contributions from different risky assets for each portfolio. The vast majority of the risk of the 60/40 portfolio comes from equities. This concentration risk is one of the motivations for the risk parity strategy, which attempts to diversify and equalize risk from different sources.

Exhibit 7 does not include any levered risk parity portfolios because, unlike utility maximization or mean-variance optimization with a specific risk aversion parameter, the risk parity strategy itself does not specify the leverage or cash position needed. In practice, however, a risk parity portfolio is often leveraged to achieve a certain return or risk target because it tends to overweight safer assets than do traditional allocations. Exhibit 8 compares these models in an alternative way by selecting leverages for the MVO and risk parity portfolios so that their volatilities will be equal to that of the 60/40 portfolio.

Not surprisingly, because the MVO and risk parity portfolios have higher Sharpe ratios than the 60/40 portfolio, their estimated returns at the same volatility levels are higher than that of the 60/40 portfolio. The levered MVO portfolio, by definition, has the highest Sharpe ratio among all portfolios in the opportunity set and therefore offers the best risk-adjusted return.

Exhibit 8: Levered MVO and risk parity portfolios with the same volatility as the 60/40 portfolio

	Asset	MVO (γ=1.7)	Risk parity	60/40
Asset allocation	Equities	42.7%	30.9%	60.0%
	Bonds	189.2%	149.1%	40.0%
	Commodities	-2.4%	26.8%	0.0%
	Cash	-129.5%	-106.7%	0.0%
Estimated annual return		3.1%	2.9%	2.8%
Estimated annual volatility		9.0%	9.0%	9.0%
Sharpe ratio		0.15	0.13	0.11
Expected maximum drawdown (1 Year)		10.2%	10.3%	10.4%

Source: PIMCO. **Hypothetical example for illustrative purposes only.**

Exhibit 9: Long-only MVO, Kelly, risk parity and traditional 60/40 portfolios

	Asset	CRRA utility maximization (MVO)			Risk parity	60/40
		γ=5	γ=2	γ=1 (Kelly)		
Asset allocation	Equities	14.0%	28.7%	50.7%	15.0%	60.0%
	Bonds	63.3%	71.3%	49.3%	72.1%	40.0%
	Commodities	0.0%	0.0%	0.0%	12.9%	0.0%
	Cash	22.8%	0.0%	0.0%	0.0%	0.0%
Risk allocation	Equities	45.2%	74.8%	96.0%	33.3%	98.4%
	Bonds	54.8%	25.2%	4.0%	33.3%	1.6%
	Commodities	0.0%	0.0%	0.0%	33.3%	0.0%
	Cash	0.0%	0.0%	0.0%	0.0%	0.0%
Estimated annual return		2.2%	2.5%	2.7%	2.3%	2.8%
Estimated annual volatility		3.0%	4.9%	7.7%	4.4%	9.0%
Sharpe ratio		0.15	0.14	0.12	0.13	0.11
Expected maximum drawdown (1 Year)		3.0%	5.2%	8.7%	4.6%	10.4%

Source: PIMCO. **Hypothetical example for illustrative purposes only.**

However, this is based on the simplifying assumption that the investor has no leverage constraints and can borrow at the risk-free rate.[8] In reality, many investors are not allowed to take leverage or have a short position for any asset. Exhibit 9 shows the MVO portfolios under the long-only constraints.

When leverage is not allowed, MVO investors who would have chosen to leverage are forced to go past the highest-Sharpe-ratio risky portfolio along the efficient frontier. The lower the risk aversion an investor has, the farther the optimal allocation will be along the frontier. This is shown in Exhibit 9. The long-only constraint also keeps the allocation to commodities at the zero lower bound. In this particular case, the 60/40 portfolio turns out to be mean-variance optimal.

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APPENDIX

Appendix 1: Comments on the functional form of the mean-variance utility

For simplicity, consider a mean-variance optimization with a risk-free asset with expected return r and a single risky asset with expected return μ and variance σ^2. Let x denote the weight in the risky asset. Then we have

$E(r_p) = (1-x)\mathrm{r} + x\mu = \mathrm{r} + x(\mu - \mathrm{r})$ and $V(r_p) = x^2\sigma^2$.

The mean-variance optimization problem is then

$$\max_x \mathrm{r} + x(\mu - \mathrm{r}) - \tfrac{1}{2}\lambda x^2 \sigma^2.$$

The first-order condition is

$$\mu - \mathrm{r} - \lambda\sigma^2 x = 0.$$

Therefore, the solution is

$$x^* = \tfrac{1}{\lambda\sigma^2}(\mu - \mathrm{r}).$$

Extension to the case of multiple risky assets is straightforward. In either case, the $\frac{1}{2}$ in the utility function leads to cleaner expressions for the first-order condition and the final solution, but more importantly, it allows λ to coincide with the risk aversion parameters in other utility functions under certain conditions.

For example, suppose the investor has a negative exponential utility over portfolio returns,

$$U(r_P) = -e^{-ar_P} \quad (a > 0),$$

where a measures the constant absolute risk aversion of the investor — i.e., $a = -\frac{U''(r_P)}{U'(r_P)}$.

If the portfolio return is normally distributed with $r_P \sim N(\mu_P, {\sigma_P}^2)$, we have

$$-U(r_P) \sim Lognormal(-a\mu_P, a^2{\sigma_P}^2).$$

By the property of lognormal distributions,

$$E[U(r_P)] = -E[-U(r_P)] = -e^{-a\mu_P + \frac{1}{2}a^2{\sigma_P}^2} = -e^{-a(\mu_P - \frac{1}{2}a{\sigma_P}^2)}.$$

Therefore, $\max E[U(r_P)] \Leftrightarrow \max \mu_P - \frac{1}{2}a{\sigma_P}^2$. This is the same functional form for the mean-variance utility, so the two risk aversion parameters, λ and a, coincide in this case.

In Appendix 2, we provide another example where λ coincides with the risk aversion parameter γ in a continuous-time CRRA utility maximization problem with lognormal prices.

Appendix 2: Continuous-time CRRA utility maximization with lognormal prices

The investor with wealth W_0 at time 0 maximizes the expected utility of final wealth at time T by choosing the optimal portfolio weights in N risky asset(s) and a risk-free asset with continuous rebalancing. The investor has no labor income or intermediate consumptions, and there are no transaction costs. The utility function is CRRA:

$$U(W) = \frac{W^{1-\gamma}-1}{1-\gamma} \qquad (\gamma > 0).$$

Assume the investment opportunities are constant over time.

Case 1: N=1

The price of the risky asset follows a geometric Brownian motion with parameters (μ,σ^2):

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t.$$

The risk-free asset follows a geometric growth rate of r:

$$\frac{dM_t}{M_t} = rdt.$$

Denote by x_t the portfolio weight in the risky asset at time t. Then $1 - x_t$ is the portfolio weight in the risk-free asset. For simplicity, assume the portfolio strategy, $\{x_t\}_{t\in[0,T]}$, is non-stochastic. For a given portfolio strategy, the dynamics of wealth is given by the following budget constraint:

$$\frac{dW_t}{W_t} = [\mathrm{r} + x_t(\mu - r)]\mathrm{dt} + x_t\sigma \mathrm{d}B_t.$$

By Itô's lemma and the budget constraint, we have

$$\mathrm{dlnW_t} = \left[\mathrm{r} + x_t(\mu - r) - \tfrac{1}{2}\sigma^2 {x_t}^2\right]\mathrm{dt} + x_t\sigma\mathrm{dB_t},$$

$$\Rightarrow lnW_T = lnW_0 + \int_0^T \left[r + x_t(\mu - r) - \tfrac{1}{2}\sigma^2 {x_t}^2\right]\mathrm{dt} + \int_0^T x_t\sigma\mathrm{d}B_t,$$

$$\Rightarrow W_T = W_0 \exp\left(\int_0^T \left[r + x_t(\mu - r) - \tfrac{1}{2}\sigma^2 {x_t}^2\right]\mathrm{dt} + \int_0^T x_t\sigma\mathrm{d}B_t\right),$$

$$\Rightarrow {W_T}^{1-\gamma} = {W_0}^{1-\gamma} \exp\left\{(1-\gamma)\int_0^T \left[r + x_t(\mu - r) - \tfrac{1}{2}\sigma^2 {x_t}^2\right]\mathrm{dt} + (1-\gamma)\int_0^T x_t\sigma\mathrm{d}B_t\right\}.$$

Because $\int_0^T x_t\sigma\mathrm{d}B_t \sim N(0, \int_0^T \sigma^2 {x_t}^2 dt)$, we have

$$E[\exp\left\{(1-\gamma)\int_0^T x_t\sigma\mathrm{d}B_t\right\}] = \exp\left\{\tfrac{1}{2}(1-\gamma)^2\int_0^T \sigma^2 {x_t}^2 dt\right\}.$$

Therefore,

$$E\left[{W_T}^{1-\gamma}\right] = {W_0}^{1-\gamma}\exp\left\{(1-\gamma)\int_0^T \left[r + x_t(\mu - r) - \tfrac{1}{2}\gamma\sigma^2 {x_t}^2\right]\mathrm{dt}\right\}.$$

Therefore, the optimal allocation solves the following problem:

$$x(.)^* = \mathrm{argmax}_{\{x_t\}_{t\in[0,T]}} E\left[\frac{{W_T}^{1-\gamma}-1}{1-\gamma}\right] = \mathrm{argmax}_{\{x_t\}_{t\in[0,T]}} \int_0^T \left[r + x_t(\mu - r) - \tfrac{1}{2}\gamma\sigma^2 {x_t}^2\right] dt.$$

The solution is a fixed weight in the risky asset at any time t:

$$x^* = \mathrm{argmax}_x \left[r + x(\mu - r) - \tfrac{1}{2}\gamma\sigma^2 x^2\right]T = \frac{\mu - r}{\gamma\sigma^2} = \frac{s}{\gamma\sigma},$$

where s is the instantaneous Sharpe ratio of the risky asset.

Therefore, the optimal wealth path satisfies

$$ln{W_T}^* = lnW_0 + \left[r + \tfrac{1}{\gamma}(1 - \tfrac{1}{2\gamma})s^2\right]T + \int_0^T \tfrac{1}{\gamma} s\mathrm{d}B_t \text{ or}$$

$$\ln\left(\frac{{W_T}^*}{W_0}\right) = \left[r + \tfrac{1}{\gamma}(1 - \tfrac{1}{2\gamma})s^2\right]T + \tfrac{1}{\gamma}s\sqrt{T}Z,$$

where $Z \sim N(0,1)$. When γ=1 (Kelly strategy), $\ln\left(\frac{{W_T}^*}{W_0}\right) = (r + \tfrac{1}{2}s^2)T + s\sqrt{T}Z.$

The probability of loss for the optimal portfolio at time T is:

$$P\left(\ln\left(\frac{{W_T}^*}{W_0}\right) < 0\right)$$

$$= P\left(\left[r + \tfrac{1}{\gamma}(1 - \tfrac{1}{2\gamma})s^2\right]T + \tfrac{1}{\gamma}s\sqrt{T}Z < 0\right)$$

$$= P\left(Z < -\frac{r + \frac{1}{\gamma}(1 - \frac{1}{2\gamma})s^2}{\frac{1}{\gamma}s}\sqrt{T}\right)$$

$$= \Phi\left(-\frac{r\gamma + (1 - \frac{1}{2\gamma})s^2}{s}\sqrt{T}\right),$$

where Φ is the cumulative distribution function (CDF) of a standard normal random variable.

Therefore, the probability of loss is decreasing in both the time horizon T and the risk aversion parameter γ.

If γ=1 (Kelly strategy), $P\left(\ln\left(\frac{{W_T}^*}{W_0}\right) < 0\right) = \Phi\left(-\frac{r + \frac{1}{2}s^2}{s}\sqrt{T}\right)$, which is increasing in s if $s^2 < 2r$ and decreasing in s if $s^2 > 2r$. Given the risk-free rate r and horizon T, the probability of loss is maximized when $s^2 = 2r$.

Case 2: $N \geq 2$

The prices of the risky assets follow a multivariate geometric Brownian motion with parameters (μ,Σ):

$$\frac{dP_t}{P_t} = \mu dt + \Gamma dB_t,$$

where $\Gamma\Gamma^T = \Sigma$.

The risk-free asset follows a geometric growth rate of r:

$$\frac{dM_t}{M_t} = rdt.$$

Denote by x_t the $N\times1$ vector of portfolio weights in the risky assets at time t. Then $(1 - x_t^T \mathbf{1})$ is the portfolio weight in the risk-free asset. For a given portfolio strategy, the dynamics of wealth is given by the following budget constraint:

$$\frac{dW_t}{W_t} = [r + x_t^{\mathrm{T}}(\mu - r)]\mathrm{d}t + x_t^T\Gamma \mathrm{d}B_t.$$

By Itô's lemma and the budget constraint, we have

$$\mathrm{dlnW_t} = \left[r + x_t^{\mathrm{T}}(\mu - r) - \tfrac{1}{2}x_t^{\mathrm{T}}\Sigma x_t\right]\mathrm{d}t + x_t^{\mathrm{T}}\Gamma \mathrm{dB_t},$$

$$\Rightarrow lnW_T = lnW_0 + \int_0^T \left[r + x_t^{\mathrm{T}}(\mu - r) - \tfrac{1}{2}x_t^T\Sigma x_t\right]\mathrm{d}t + \int_0^T x_t^T\Gamma \mathrm{d}B_t,$$

$$\Rightarrow W_T = W_0 \exp\left(\int_0^T \left[r + x_t^{\mathrm{T}}(\mu - r) - \tfrac{1}{2}x_t^T\Sigma x_t\right]\mathrm{d}t + \int_0^T x_t^T\Gamma \mathrm{d}B_t\right),$$

$$\Rightarrow W_T^{1-\gamma} = W_0^{1-\gamma} \exp\left\{(1-\gamma)\int_0^T \left[r + x_t^{\mathrm{T}}(\mu - r) - \tfrac{1}{2}x_t^T\Sigma x_t\right]\mathrm{d}t + (1-\gamma)\int_0^T x_t^T\Gamma \mathrm{d}B_t\right\}.$$

Because $\int_0^T x_t^T\Gamma \mathrm{d}B_t \sim N(0,\ \int_0^T x_t^T\Sigma x_t dt)$, we have

$$E[\exp\left\{(1-\gamma)\int_0^T x_t^T\Gamma \mathrm{d}B_t\right\}] = \exp\left\{\tfrac{1}{2}(1-\gamma)^2\int_0^T x_t^T\Sigma x_t dt\right\}.$$

Therefore,

$$E\left[W_T^{1-\gamma}\right] = W_0^{1-\gamma} \exp\left\{(1-\gamma)\int_0^T \left[r + x_t^{\mathrm{T}}(\mu - r) - \tfrac{1}{2}\gamma x_t^T\Sigma x_t\right]\mathrm{d}t\right\}.$$

Therefore, the optimal allocation solves the following problem:

$$x(.)^* = \mathrm{argmax}_{\{x_t\}_{t\in[0,T]}} E\left[\frac{W_T^{1-\gamma}-1}{1-\gamma}\right] = \mathrm{argmax}_{\{x_t\}_{t\in[0,T]}} \int_0^T \left[r + x_t^{\mathrm{T}}(\mu - r) - \tfrac{1}{2}\gamma x_t^T\Sigma x_t\right]dt.$$

The solution is a fixed vector of weights in the risky assets at any time t:

$$x^* = \underset{x}{argmax}\left[r + x^T(\mu - r) - \frac{1}{2}\gamma x^T\Sigma x\right]T = \frac{1}{\gamma}\Sigma^{-1}(\mu - r).$$

If N=2, we can invert $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ and write the optimal weights to the two risky assets as functions of their volatilities, Sharpe ratios and correlation ρ:

$$x^* = \frac{1}{\gamma(1-\rho^2)}\begin{pmatrix} \frac{1}{\sigma_1}(s_1 - \rho s_2) \\ \frac{1}{\sigma_2}(s_2 - \rho s_1) \end{pmatrix}.$$

Appendix 3: Mean-variance optimality of risk parity

Below we show one way to prove the statement, and there are potentially many other ways (see, for example, Maillard, Roncalli and Teiletche 2010).

Define the following additional notations:

$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)^T$ is the $N\times1$ vector of volatilities of the risky assets;

$D=diag(\sigma)$ is the $N\times N$ diagonal matrix with the volatilities as its diagonal elements;

C is the correlation matrix of the risky assets — i.e., $\Sigma=DCD$; $\mathbf{1} = (1, 1, \ldots, 1)^T$.

Assume further:

1. All risky assets have the same Sharpe ratios: $\mu - r = k\sigma$, where k is a constant.
2. All the pairwise correlations are the same: $\rho_{ij} = \rho, \forall\, i \neq j$.

Assumption 2 implies

$$C\mathbf{1} = [1 + (N-1)\rho]\mathbf{1}.$$

Premultiplying both sides by C^{-1}, we have

$$C^{-1}\mathbf{1} = \frac{1}{1+(N-1)\rho}\mathbf{1}.$$

Therefore,

$$Dx^* = D\left[\frac{1}{\gamma}\Sigma^{-1}(\mu - r)\right] = \frac{1}{\gamma}D(D^{-1}C^{-1}D^{-1})(k\sigma) = \frac{k}{\gamma}C^{-1}\mathbf{1} = \frac{k}{\gamma[1+(N-1)\rho]}\mathbf{1},$$

which means $x_i^*\sigma_i = \frac{k}{\gamma[1+(N-1)\rho]}, \forall\, i$.

The MVO portfolio therefore is a risk parity portfolio in the sense of both equal stand-alone risks and equal risk contributions (the two coincide under the assumption of identical pairwise correlations).

Appendix 4: Probability that the Kelly strategy will outperform another fixed mix strategy

Consider the following simple example with one risky asset.

The price of the risky asset follows a geometric Brownian motion with parameters (μ, σ^2):

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t.$$

The risk-free asset follows a geometric growth rate of r:

$$\frac{dM_t}{M_t} = rdt.$$

Denote by x_t the portfolio weight in the risky asset at time t. Then $1-x_t$ is the portfolio weight in the risk-free asset. For a given strategy, the dynamics of wealth is given by the following budget constraint:

$$\frac{dW_t}{W_t} = [r + x_t(\mu - r)]\mathrm{d}t + x_t\sigma \mathrm{d}B_t.$$

By Itô's lemma and the budget constraint, we have

$$\mathrm{dlnW_t} = \left[r + x_t(\mu - r) - \frac{1}{2}\sigma^2 x_t^2\right]\mathrm{d}t + x_t\sigma\mathrm{dB_t}$$

$$\Rightarrow lnW_T = lnW_0 + \int_0^T \left[r + x_t(\mu - r) - \frac{1}{2}\sigma^2 x_t^2\right]\mathrm{d}t + \int_0^T x_t\sigma\mathrm{d}B_t$$

$$\Rightarrow E[lnW_T] = lnW_0 + \int_0^T \left[r + x_t(\mu - r) - \tfrac{1}{2}\sigma^2 x_t^2\right]\mathrm{d}t.$$

To maximize $E[lnW_T]$, the Kelly strategy selects $x_t^K = x^K = \frac{\mu - r}{\sigma^2}$, which maximizes

$$f(x) \equiv r + x(\mu - r) - \frac{1}{2}\sigma^2 x^2.$$

Now suppose there is an alternative fixed mix strategy: $x_t^A = x^A \neq x^K$. We can state the wealth under the two strategies at time T as:

$$W_T^K = W_0 \exp\left\{\left[r + x^K(\mu - r) - \frac{1}{2}{x^K}^2\sigma^2\right]T + x^K\sigma\sqrt{T}Z\right\},$$

$$W_T^A = W_0 \exp\left\{\left[r + x^A(\mu - r) - \frac{1}{2}{x^A}^2\sigma^2\right]T + x^A\sigma\sqrt{T}Z\right\},$$

where $Z \sim N(0,1)$.

The probability that the Kelly strategy will outperform the alternative strategy at time T:

$$P(W_T^K > W_T^A)$$

$$= P\left(\left[r + x^K(\mu - r) - \frac{1}{2}{x^K}^2\sigma^2\right]T + x^K\sigma\sqrt{T}Z > \left[r + x^A(\mu - r) - \frac{1}{2}{x^A}^2\sigma^2\right]T + x^A\sigma\sqrt{T}Z\right)$$

$$= \begin{cases} P\left(Z > \frac{f(x^A) - f(x^K)}{(x^K - x^A)\sigma}\sqrt{T}\right) = \Phi\left(\frac{f(x^K) - f(x^A)}{(x^K - x^A)\sigma}\sqrt{T}\right) & if\ x^K > x^A \\ P\left(Z < \frac{f(x^A) - f(x^K)}{(x^K - x^A)\sigma}\sqrt{T}\right) = \Phi\left(\frac{f(x^A) - f(x^K)}{(x^K - x^A)\sigma}\sqrt{T}\right) & if\ x^K < x^A \end{cases},$$

where Φ is the cumulative distribution function (CDF) of a standard normal random variable.

Because x^K is the unique solution to $\max_x f(x)$, we have $f(x^K) > f(x^A), \forall\, x^A \neq x^K$.

Therefore, $P(W_T^K > W_T^A) > 0.5, \forall\, T > 0$ and $\lim_{T\to\infty} P(W_T^K > W_T^A) = 1$.

Appendix 5: Estimation of EMDD

Suppose the value of a portfolio follows a geometric Brownian motion with parameters $(\mu_P, {\sigma_P}^2)$:

$$\frac{dW_t}{W_t} = \mu_P dt + \sigma_P dB_t.$$

Define EMDD as the expected percentage loss (approximated by the absolute log return) from the peak to the trough over a specific time horizon T. Based on Magdon-Ismail et al. (2004), we have

$$EMDD = \begin{cases} \frac{2{\sigma_P}^2}{\mu_P - \frac{1}{2}{\sigma_P}^2} Q_p\left(\frac{\left(\mu_P - \frac{1}{2}{\sigma_P}^2\right)^2}{2{\sigma_P}^2}T\right) & if\ \mu_P - \frac{1}{2}{\sigma_P}^2 > 0 \\ 1.2533\sigma_P\sqrt{T} & if\ \mu_P - \frac{1}{2}{\sigma_P}^2 = 0, \\ -\frac{2{\sigma_P}^2}{\mu_P - \frac{1}{2}{\sigma_P}^2} Q_n\left(\frac{\left(\mu_P - \frac{1}{2}{\sigma_P}^2\right)^2}{2{\sigma_P}^2}T\right) & if\ \mu_P - \frac{1}{2}{\sigma_P}^2 < 0 \end{cases}$$

where Q_p and Q_n are functions tabulated in that paper.

If the portfolio follows the wealth path resulting from a CRRA utility maximization problem, as in Appendix 2 (Case 1), $\mu_P = r + x^*(\mu - r) = r + \frac{s^2}{\gamma}$ and $\sigma_P = x^*\sigma = \frac{s}{\gamma}$, where r is the risk-free rate, s is the instantaneous Sharpe ratio of the risky asset, and γ is the risk aversion parameter. Therefore, we can estimate EMDD for optimal strategies under different risk aversion parameters by plugging in the model assumptions.

Appendix 6: Relationship between mean log return and mean arithmetic return

Taking the second-order Taylor expansion of ln $(1+r)$ around $E(r)$, we have

$$\ln(1 + r) \approx \ln(1 + E(r)) + \frac{1}{1 + E(r)}(r - E(r)) - \frac{1}{2(1 + E(r))^2}(r - E(r))^2.$$

Taking expectations on both sides, we have

$$E(\ln(1 + r)) \approx \ln(1 + E(r)) - \frac{1}{2(1 + E(r))^2}Var(r).$$

If $E(r)$ is small (such as when the time interval is short), $\ln(1+E(r)) \approx E(r)$. Therefore, $E(\ln(1 + r)) \approx E(r) - \frac{1}{2(1 + E(r))^2}Var(r)$.

It says an asset's expected log return is lower than its expected arithmetic return by approximately half of the variance of the arithmetic return. This relationship holds exactly in continuous time when the asset price follows a geometric Brownian motion:

$$r \equiv \frac{dP_t}{P_t} = \mu dt + \sigma dB_t.$$

By Itô's lemma,

$$dlnP_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t.$$

Therefore,

$$E(dlnP_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt.$$

[1] For curious readers, this specific functional form serves two purposes. Apart from allowing for cleaner expressions for the first-order condition and the solution, it ensures that the risk aversion parameter in the mean-variance utility coincides with the risk aversion parameters in some other popular utility functions under certain conditions. Appendix 1 provides a detailed example.

[2] Continuous rebalancing is the limiting case of small rebalancing intervals. It makes it easy or possible to derive analytical solutions that provide economic insights difficult to obtain from numerical solutions and allows us to distinguish general properties of the solutions from those relying on specific parameter values. Continuous rebalancing can be a good approximation for frequently rebalanced portfolios, especially when the investment horizon is long.

[3] To see why the CRRA utility is reduced to a log utility when γ=1, first note that $W^{1-\gamma} = e^{(1-\gamma)\log(W)}$. Then we can apply L'Hôpital's rule and take derivatives of both the numerator and the denominator: $\lim_{\gamma\to 1}\frac{e^{(1-\gamma)\log(W)}-1}{1-\gamma} = \lim_{\gamma\to 1}\frac{-\log(W)W^{1-\gamma}}{-1} = \log(W)$.

[4] This idea was shared later by other researchers and practitioners (see, for example, Ziemba and Ziemba 2007).

[5] Roncalli (2014) showed risk parity for factors is equivalent to risk budgeting for asset classes with a specific risk budget profile.

[6] See Frazzini and Pedersen (2014) for more details.

[7] The 60/40 mix appealed to many investors for decades because of its simplicity and the risk/return trade-off it delivered. With the fixed target weights, long-term investors do not have to time the market. The regular rebalancing to the fixed weight may also potentially contribute to the performance of the strategy in the presence of mean reversion for asset returns.

[8] Another caveat is that we use volatility as the measure of risk here, and it is limited in capturing any tail risk beyond what is implied by normal distributions.

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Past performance is not a guarantee or a reliable indicator of future results.

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650 Newport Center Drive
Newport Beach, CA 92660
+1 949.720.6000

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